A Dynamical Theory of Quantum Measurement and Spontaneous Localization

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Abstract

We develop a rigorous treatment of discontinuous stochastic unitary evolution for a system of quantum particles that interacts singularly with quantum "bubbles" at random instants of time. This model of a "cloud chamber" allows to watch and follow with a quantum particle trajectory like in cloud chamber by sequential unsharp localization of spontaneous scatterings of the bubbles. Thus, the continuous reduction and spontaneous localization theory is obtained as the result of quantum filtering theory, i.e., a theory describing the conditioning of the a priori quantum state by the measurement data. We show that in the case of indistinguishable particles the a posteriori dynamics is mixing, giving rise to an irreversible Boltzmann-type reduction equation. The latter coincides with the nonstochastic Schrödinger equation only in the mean field approximation, whereas the central limit yields Gaussian mixing fluctuations described by stochastic reduction equations of diffusive type.

1 Introduction

The quantum measurement theory based on the ordinary von Neumann reduction postulate applies neither to instantaneous observations with continuous spectra nor to continual (continuous in time) measurements. Although such phenomena can be described in the more general framework of Ludwig's Davies-Lewies operational approach [1]–[5], there is a particular interest in describing quantum measurements by concrete Hamiltonian models from which the operational description can be derived by an averaging procedure. Perhaps the first model of such kind for instantaneous unsharp measurement of particle localization was given by von Neumann [6]. He considered the singular interaction

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Hamiltonian

$$h_x(t) = x\delta(t)\frac{\hbar}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}y}, \quad \delta(t) = \begin{cases} \infty, & t = 0, \\ 0, & t \neq 0 \end{cases} \tag{1.1}$$

with Dirac δ-function, producing the translation operator

$$s_x = \exp\left\{ -\frac{\mathrm{i}}{\hbar}\int_{-\infty}^{\infty} h_x(t)\,\mathrm{d}t \right\} = e^{-x\mathrm{d}/\mathrm{d}y} \tag{1.2}$$

at time $t = 0$ of the measurement. Here x is the position of the particle and y is the pointer position given by the q-coordinate of a quantum meter. The particle scattering operator $S = \{s_x\}$ applied to the (generalized) position eigen-vectors $|x\rangle$ as $S|x\rangle = |x\rangle s_x$ does not affect the position x of the particle but changes the meter coordinate q to $y = s_x^\dagger q s_x = x + q$. This implies that the initial wave function $\psi_0(x, y)$ of the system "particle plus meter" is transformed into

$$\psi(x, y) = s_x\psi_0(x, y) = \psi_0(x, y - x). \tag{1.3}$$

If in the initial state the particle and the meter were not coherent, $\psi_0(x, y) = \eta(x)f_0(y)$, and the wave function f_0 of the meter was fixed, then one can obtain the unitary transformation $S : \psi_0 \mapsto s_\bullet\psi_0$ via a family $\{F(y)\}$ of reduction transformations $F(y) : \eta \mapsto f_\bullet(y)\eta$ for the particle vector-states η. Specifically, Eq. (1.3) can be defined as

$$\psi(x, y) = f_0(y - x)\eta(x) \equiv f_x(y)\eta(x). \tag{1.4}$$

The linear nonunitary operators $F(y) = s_\bullet f_0(y)$ act on $|x\rangle$ as the multiplication $F(y)|x\rangle = |x\rangle f_x(y)$ by $f_x(y) = f_0(y - x) = s_x f_0(y)$ and would give a sharp localization of any particle wave function $\eta(x)$ at the point $x = y$ of the pointer position provided that the wave function $f_0(y)$ could be initially localized at $y = 0$. But there are no such sharply localized quantum states for the continuous pointer, and the best that one can do is to take a wave packet $f_0(y)$, say, of the Gaussian form

$$f_0(y) = \exp\left\{ -\frac{\pi}{2}y^2 \right\}. \tag{1.5}$$

This results in the unsharp localization

$$\psi(x, y) = \exp\left\{-\frac{\pi}{2}(x - y)^2\right\}\eta(x) \tag{1.6}$$

of any particle wave function $\eta(x)$ about the observed value y of the pointer, normalized to the probability density

$$\mathrm{p}_0(y) = \int |f_0(y - x)|^2|\eta(x)|^2 \; x\,. \tag{1.7}$$

For commuting operators x and y, this is equivalent to the classical measurement model $y = x + q$ for unsharp measurement of an unknown signal x via the sharp

measurement of the signal plus Gaussian noise q with given probability density $|f_0(q)|^2$.

These arguments illustrate how to interpret the reduction model involving the continuous spectrum of a quantum measurement as a Hamiltonian interaction model with nondemolition observation for a quantum object using the measurement of the pointer coordinate of the quantum meter. Since von Neumann introduced this approach, it was used by numerous other authors [7, 8] for the derivation of a generalized reduction $\eta \mapsto F(y)\eta$ that would replace the von Neumann postulate $\eta \mapsto E(y)\eta$ given by orthoprojections $\{E(y)\}$. As extended to nondemolition observations continual in time [9]–[15], this approach consists in using the quantum filtering method for the derivation of nonunitary stochastic wave equations describing the quantum dynamics under the observation. Since a particular type of such equations has been taken as a postulate in the phenomenological theory of continuous reduction and spontaneous localization [16]–[20], the question arises whether it is possible to obtain this equation from an appropriate Schrödinger equation. Here we shall show how this can be done by second quantization of the interaction Hamiltonian considered by von Neumann, obtaining a stochastic model of continual nondemolition observation for the position of a quantum particle by counting some other quanta. But first we show that even the projection postulate can be derived in the framework of this approach with the suggested Hamiltonian interaction and a proper nondemolition observation.

2 Hamiltonian Reduction Model

Let $\mathcal{H}$ be a Hilbert space, called the state space of a particle, and let $\mathbf{R} = \{R^\alpha : \alpha = 1, \ldots, d\}$ be selfadjoint operators in $\mathcal{H}$ with either integer or continuous spectrum $\mathbb{Z}^d$ or $\mathbb{R}^d$, $d \in \{1, 2, 3\}$. Let $\kappa > 0$ be a scaling parameter. If the operators commute, one can regard the scaled vector operator $\kappa\mathbf{R}$ as the position $\mathbf{x}$ of the particle in $\mathbb{R}^d$ or in the d-dimensional lattice $\kappa\mathbb{Z}^d$ if it is quantized. In the $\mathbf{x}$-representation, the operator acts as the multiplication $\kappa R^\alpha|\mathbf{x}\rangle = |\mathbf{x}\rangle\lambda(x^\alpha)$ by $\lambda(x^\alpha) = \kappa\lfloor x^\alpha/\kappa\rfloor$, where $\lfloor\mathbf{x}\rfloor \in \mathbb{Z}^d$ denotes the integer part of the vector $\mathbf{x} = \{x^\alpha : \alpha = 1, \ldots, d\}$.

A quantum meter with continuous ($\Lambda = \mathbb{R}^d$) or lattice ($\Lambda = \varepsilon\mathbb{Z}^d$) pointer scale is described by the Hilbert space $L^2(\Lambda)$ of complex-valued functions $f : \Lambda \to \mathbf{C}$ square integrable in the sense that $\|f\|^2 = \int |f(\mathbf{y})|^2 \mathrm{d}\lambda < \infty$, where

$$\int f(\mathbf{y})\,\mathrm{d}\lambda = \begin{cases} \sum_{y\in\Lambda} f(\mathbf{y})\varepsilon^d\,, & \text{if } \Lambda = \varepsilon\mathbb{Z}^d\,, \quad \mathrm{d}\lambda = \varepsilon^d \\ \int_{\mathbf{y}\in\Lambda} f(\mathbf{y})\mathrm{d}\mathbf{y}\,, & \text{if } \Lambda = \mathbb{R}^d\,, \quad \mathrm{d}\lambda = \mathrm{d}y\,. \end{cases}.$$

Consider a moving particle with Hamiltonian H in $\mathcal{H}$. Its singular evolution corresponding to the position measurement at time $t = 0$ is described in the product space $\mathcal{H}_1 = \mathcal{H} \otimes L^2(\Lambda)$ by the time-dependent Hamiltonian

$$H_1(t) = H_0 + \kappa\mathbf{R} \otimes \delta(t)\mathbf{P}. \tag{2.1}$$

Here $H_0 = H \otimes \mathbf{1}$, where $\mathbf{1}$ is the identity operator in $L^2(\Lambda)$, and $\mathbf{R} \otimes \mathbf{P} \equiv \sum P_\alpha R^\alpha$, where P_α, $\alpha = 1, \ldots, d$ are operator components of the meter momentum vector $\mathbf{P} = (P_1, \ldots, P_d)$ with the pointer coordinate vector $\mathbf{q} = \{q^\alpha : \alpha = 1, \ldots, d\}$ given by the multiplication operators $(q^\alpha f)(\mathbf{y}) = y^\alpha f(\mathbf{y})$, $\mathbf{y} \in \Lambda$ in the Hilbert space $L^2(\Lambda)$ of its quantum states such that

$$\|\mathbf{q}f\|^2 = \int \sum_{\alpha=1}^{d} |y^\alpha f(y)|^2 \mathrm{d}\lambda < \infty \,.$$

The operators $P_\alpha = -\mathrm{i}\hbar\partial/\partial y^\alpha$ have the bounded spectrum $(\pi\hbar)[-1/\varepsilon, 1/\varepsilon)^d$ for $\Lambda = \varepsilon\mathbb{Z}^d$ and are defined by the matrix elements

$$(\mathbf{y}|P_\alpha|\mathbf{q}) = \frac{1}{(2\pi\hbar)^d} \int p_\alpha e^{\mathrm{i}(\mathbf{y}-\mathbf{q})\mathbf{p}/\hbar} \mathrm{d}\mathbf{p} \,, \quad \mathbf{q} \in \Lambda \,,$$

where $(\mathbf{y}|\mathbf{q}) = \delta(\mathbf{y} - \mathbf{q})$, $\mathbf{y} \in \Lambda$, are the (generalized) eigenfunctions of the coordinate operators q^α normalized with respect to the measure $\mathrm{d}\lambda = \varepsilon^d$ in the discrete case $y^\alpha \in \varepsilon\mathbb{Z}$. This generates the shift operators (1.2) in $L^2(\Lambda)$ with $\mathbf{x} \in \Lambda$ and the scattering operator

$$S_t = \exp\left(-\frac{\mathrm{i}}{\hbar}\kappa \mathbf{R} \otimes \mathbf{P}_t\right) = \begin{cases} S \,, & t > 0 \\ I \,, & t \leqslant 0 \end{cases} \tag{2.2}$$

in $\mathcal{H}_1$, where $\mathbf{P}_t = 1_t \mathbf{P}$, $1_t = 1$ if $t > 0$, and $1_t = 0$ otherwise.

The singular time dependence of the Hamiltonian (2.1) makes it impossible to define the Schrödinger equation $\mathrm{i}\hbar \mathrm{d}\psi/\mathrm{d}t = H_1(t)\psi(t)$ in the usual sense. But one can define a unitary stochastic evolution $U_1(t) : \mathcal{H}_1 \to \mathcal{H}_1$ for some $t_0 < 0$ as the single-jump unitary process

$$U_1(t) = \exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} H_1(s)\mathrm{d}s\right) = e^{\mathrm{i}H_0(t_0 - t)/\hbar} S_t$$

provided that $[R^a, H] = 0$. Usually the commutator is not zero, and the unitary evolution corresponding to (2.1) must be redefined in terms of the solution $\psi(t) = U_1(t)\psi_0$ to the regularized wave equation. This can be done in terms of the forward differentials

$$\mathrm{d}\psi(t) = \psi(t + \mathrm{d}t) - \psi(t), \quad \mathrm{d}1_t = 1_{t+\mathrm{d}t} - 1_t \,;$$

namely, we define the generalized Schrödinger equation as

$$\mathrm{d}\psi(t) + \frac{\mathrm{i}}{\hbar} H_0 \psi(t)\mathrm{d}t = (S - I)\psi(t)\mathrm{d}1_t \,, \quad \psi(t_0) = \psi_0 \,. \tag{2.3}$$

Proposition 1 *For every $\psi_0 \in \mathcal{H}$ the difference equation (2.3) has a unique solution corresponding to the initial value $t_0 \leqslant 0$; this solution is given by the unitary operator*

$$U_1(t) = U_0(t - t_0) S_t(-t_0), \quad \textit{where} \;\; U_0(t) = \exp\{-\mathrm{i}H_0 t/\hbar\}, \; S_t(r) = U_0^\dagger(r) S_t U_0(r).$$

Proof. Let us rewrite the generalized Schrödinger equation in the integral form

$$\psi(t) = e^{-\mathrm{i}H_0 t/\hbar}\left(e^{\mathrm{i}H_0 t_0/\hbar}\psi_0 + \int_{t_0}^{t} e^{\mathrm{i}H_0 r/\hbar}(S-I)\psi(r)\mathrm{d}1_r\right) \tag{2.4}$$

(the equivalence of (2.4) to (2.3) can be shown by straightforward differentiation). We can write $\psi(0) = U_0(-t_0)\psi_0$ for the solution (2.4) when $t = 0$ since $\int_{t_0}^{t}\psi(r)\mathrm{d}1_r = 0$ for any $t_0 \leqslant 0$; hence $\psi(t)$ can be rewritten as

$$\psi(t) = U_0(t)(\psi(0) + (S-I)1_t\psi(0)) = U_0(t)S_t\psi(0)$$

for any $t_0 < 0$ since

$$\int_{t_0}^{t}\psi(r)\mathrm{d}1_r = 1_t\psi(0)$$

and $(S-I)1_t = S_t - I$ by the definition of the scattering operator (2.2). Thus, $U(t)$ is the unitary operator

$$U_0(t)S_tU_0(-t_0) = U_0(t-t_0)S_t(-t_0).$$

This gives the solution to equation (2.3) for $t_0 = 0$ as well, since $\psi(t) = U_0(t)S_t\psi(0)$ is equal to ψ_0 for $t = 0$. ∎

The rescaled pointer $\mathbf{q}_t = 1_t\mathbf{q}$ switched on at the instant $t = 0$ of the scattering is described in the space $\mathcal{H}\otimes L^2(\Lambda)$ by the operators $\mathbf{Q}_t = \kappa^{-1}I\otimes\mathbf{q}_t$. In the Heisenberg picture $U_1^\dagger(t)\mathbf{Q}_tU_1(t)$ these operators are described for any $t_0 < 0$ by the operators $\mathbf{Y}_t(r) = S^\dagger(r)\mathbf{Q}_tS(r)$ taken at $r = -t_0$; for all $t > 0$ these operators have the shifted form

$$\mathbf{Y}_t(r) = \mathbf{R}(r)\otimes\mathbf{1}_t + \frac{1}{\kappa}I\otimes\mathbf{q}_t \equiv \mathbf{R}_t(r) + \mathbf{Q}_t\,, \tag{2.5}$$

where $\mathbf{R}(r) = U^\dagger(r)\mathbf{R}U(r)$ and $\mathbf{1}_t$ is the operator $\mathbf{1}$ for $t > 0$ and $\mathbf{0}$ for $t \leqslant 0$ in $L^2(\Lambda)$. It follows from the commutativity condition

$$[Y_s^\alpha, Y_t^\beta] = 0\,, \quad \forall\, s,t;\ \ \alpha,\beta = 1,\ldots,d \tag{2.6}$$

that the observables $\{\mathbf{Y}_t\}$ are selfnondemolition in the sense of their joint measurability, and they are nondemolition with respect to an arbitrary particle operator $X_t(r) = U_1(t)^\dagger(X\otimes\mathbf{1})U_1(t)$ at $t_0 = -r$ in the Heisenberg picture in the sense of their predictability [9, 13]

$$[X_s, Y_t^\alpha] = 0\,, \quad \forall\, s \geqslant t;\ \ \alpha = 1,\ldots,d;$$

indeed, $[X_s, Y_s^\alpha] = 0$ and $Y_s = Y_t$ for $s,\, t > 0$ since

$$U_1^\dagger(s)\mathbf{Q}_tU_1(s) = S^\dagger(-t_0)\mathbf{Q}_tS(-t_0)\,, \quad \forall\, s \geqslant t\,.$$

Let us fix a state vector $f_0 \in L^2(\Lambda)$, $\|f_0\| = 1$, given by a localized wave function $f_0(\mathbf{y})$ on Λ at $\mathbf{y} = 0$. Let $|\mathbf{y})$ denote the (generalized) eigenfunction

in the spectral representation $\mathbf{q} = \int \mathbf{y}|\mathbf{y})(\mathbf{y}|\mathrm{d}\lambda$. In the discrete case one can take the sharply localized $f_0 = \varepsilon^{d/2}|0)$ given by the function $f_0(\mathbf{y}) = e(\mathbf{y})/\varepsilon^{d/2}$, where $e(\mathbf{y}) = 1$ if $\mathbf{y} = 0$ and $e(\mathbf{y}) = 0$ if $\mathbf{y} \neq 0$. This yields the localizing transformations $F_t(\mathbf{y}) = (\mathbf{y}|S_t f_0$ in the form

$$F_t(\mathbf{y}) = f_0(\mathbf{y}I - \mathbf{R}_t) = \begin{cases} f_0(\mathbf{y}I - \mathbf{R})\,, & t > 0 \\ f_0(\mathbf{y})I\,, & t \leqslant 0. \end{cases}$$

The reduced transformations $\eta \mapsto \psi(t, \mathbf{y})$, $\mathbf{y} \in \Lambda$, defined on the particle space $\mathcal{H}$ by the formula

$$\psi(t, \mathbf{y}) = U(t - t_0)F_t(-t_0, \mathbf{y})\eta\,, \quad \mathbf{y} \in \Lambda,$$

with $U(t) = \exp\{-\mathrm{i}/\hbar H_t\}$ and $F_t(r) = U^\dagger(r)F_tU(r)$ reproduce the unitary evolution $U_1(t)$ on $\eta \otimes f_0 \in \mathcal{H}_1$ similarly to Eq. (1.3) and (1.4), namely,

$$\psi(t, \mathbf{y}) = U(t)(\mathbf{y}|S_t f_0 U(-t_0)\eta = (\mathbf{y}|U_1(t)(\eta \otimes f_0)\,, \quad \forall\, \eta \in \mathcal{H}\,.$$

The operators $F_t(r)$ at $r = -t_0$ with an initial wave function η of the particle before the scattering ($t_0 \leqslant 0$) define the probability measure

$$\mu_t(\Delta) = \int_\Delta \|F(t, \mathbf{y})\eta\|^2 \mathrm{d}\lambda = \langle \eta, \Pi_t(\Delta)\eta\rangle\,, \quad \Delta \subseteq \Lambda,$$

for the statistics of the nondemolition measurement of $\kappa\mathbf{R}$ via the observation of the pointer position $\mathbf{y} \in \Delta$ after the scattering. It is given by a positive operator-valued measure $\Pi_t(\Delta) = \int_\Delta |f_0(\mathbf{y} - \kappa\mathbf{R}_t(-t_0))|^2 \mathrm{d}\lambda$, which is normalized, $\Pi_t(\Lambda) = I$, since the measure $\mathrm{d}\lambda$ on $\varepsilon\Lambda = \mathbb{Z}^d$ or $\Lambda = \mathbb{R}^d$ is translation-invariant.

By renormalizing the operators $F(\mathbf{q})$ as $E(\mathbf{y}) = \varepsilon^{d/2}F(\mathbf{y})$ with step $\varepsilon = \kappa$, one obtains the orthogonal projections

$$E(\mathbf{y}) = e(\kappa\mathbf{R} - \mathbf{y}I) = \int_{\mathbf{x}:\lfloor \mathbf{x}/\kappa \rfloor = \mathbf{y}/\kappa} |\mathbf{x}\rangle\langle\mathbf{x}|\mathrm{d}\mathbf{x}\,.$$

in the case $\kappa\mathbf{R} = \int \lambda(\mathbf{x})|\mathbf{x}\rangle\langle\mathbf{x}|\mathrm{d}\mathbf{x}$ when $\lambda(\mathbf{x}) = \kappa\lfloor \mathbf{x}/\kappa \rfloor$. Thus, $\Pi_t(\Delta)$, $\Delta \subseteq \Lambda$, is the spectral measure $\sum_{y \in \Delta} E(t_0, \mathbf{y})$ of the quantized position $\kappa\mathbf{R}(-t_0)$ of the particle for a $t > 0$ given by the eigenorthoprojections $E(r, \mathbf{y}) = U^\dagger(r)E(\mathbf{y})U(r)$ of the operators $\mathbf{R}(r)$, corresponding to the rescaled pointer integer values $\mathbf{y}/\kappa$. Thus, the projection reduction postulate has been deduced >from the Hamiltonian interaction (2.1) and the nondemolition measurement for the sharply localized initial state f_0. But there is no continuous limit as $\kappa \to 0$ of such sharp reduction with nontrivial $e \neq 0$, since $\|e\|^2 = \int e(\mathbf{y})\mathrm{d}\lambda = \kappa^d \to 0$ and the sharp function $e(y)$ disappears as the element of the meter state space $L^2(\Lambda)$.

In the continuous case, one can take an unsharp $f_0 \in \Lambda^2(\Lambda)$ and renormalize the operators (**??**) as $G_t(\mathbf{y}) = f_0(\mathbf{y}I - \kappa\mathbf{R}_t)/f_0(\mathbf{y})$ if $f_0(\mathbf{y}) \neq 0$, as in the

Gaussian case (1.5). They define $G_t(\mathbf{y})$ as the identity operator for $t \leqslant 0$ or $r > 0$, whereas for $t > 0$ and $r \leqslant 0$ the operator

$$G(\mathbf{y}) = \langle \mathbf{y}|Sf_0 = (\mathbf{y}|G\,, \quad G = f_0^{-1}Sf_0\,, \tag{2.7}$$

say, of the Gaussian form

$$G(\mathbf{y}) = \exp\{\pi\kappa\mathbf{R}(\mathbf{y}I - \frac{1}{2}\,\kappa\mathbf{R})\}$$

given by the generalized eigenfunctions $|\mathbf{y}\rangle = |\mathbf{y})/f_0(\mathbf{y})$ for the spectral representation $\mathbf{q} = \int \mathbf{y}|\mathbf{y}\rangle\langle\mathbf{y}|\mathrm{d}\mu_0$ with respect to the initial probability measure $\mathrm{d}\mu_0 = |f_0(\mathbf{y})|^2\mathrm{d}\lambda$ with density $|f_0(\mathbf{y})|^2 = \exp\{-\pi\mathbf{y}^2\}$ in the case (1.5). The corresponding propagators

$$T(t,\mathbf{y}) = U(t-t_0)G_t(-t_0,\mathbf{y}), \quad \mathbf{y} \in \Lambda,\ t_0 \leqslant 0$$

define the operator-valued measure $\Pi_t(\Delta)$ as

$$\Pi_t(\Delta) = \int_\Delta T^\dagger(t,\mathbf{y})T(t,\mathbf{y})\mathrm{d}\mu_0 = \int_\Delta |G_t(-t_0,\mathbf{y})|^2\mathrm{d}\mu_0,$$

so that the output probability measure $\mu_t(\Delta) = \langle\eta, \Pi_t(\Delta)\eta\rangle$ is absolutely continuous with respect to μ_0, $\mu_0(\Delta) = 0 \Rightarrow \mu_t(\Delta) = 0$. Hence, the reduced state vector $\chi(t,\mathbf{y}) = T(t,\mathbf{y})\eta$ is normalized to 1 as a stochastic vector process $\chi(t) : \mathbf{y} \mapsto \chi(t,\mathbf{y}) \in \mathcal{H}$ in the mean square sense with respect to the input probability measure μ_0,

$$\|\chi(t)\|_0^2 = \int \langle\chi(t,\mathbf{y}), \chi(t,\mathbf{y})\rangle\mathrm{d}\mu_0 = \langle\eta,\eta\rangle = 1\,.$$

This model of nondemolition observation with continuous data $\mathbf{y} \in \mathbb{R}^d$ also applies to unsharp measurement of operators $\mathbf{R}$ with discrete spectrum. In contrast to sharp measurement, unsharp measurement is not sensitive to the continuous spectrum limit as $\kappa \to 0$ of $\mathbf{R} = \lfloor\mathbf{x}/\kappa\rfloor$, corresponding to the replacement of $\kappa\mathbf{R}$ by $\mathbf{x} \in \mathbb{R}^d$.

Theorem 1 *For any initial $t_0 \leqslant 0$ the stochastic vector process $\chi(t) = T(t)\eta$ satisfies the single-kick equation*

$$\mathrm{d}\chi(t) + \frac{\mathrm{i}}{\hbar}H\chi(t)\mathrm{d}t = \mathrm{d}1_t[G-I]\chi(t)\,, \quad \chi(t_0) = \eta,$$

generated by the random differential $\mathrm{d}1_t[G-I](\mathbf{y}) = (G(\mathbf{y})-I)\mathrm{d}1_t$ on $\mathbf{y} \in \Lambda$ with respect to the initial probability measure μ_0. This simplest reduction equation is written in terms of the forward differentials $\mathrm{d}\chi(t,\mathbf{y}) = \chi(t+\mathrm{d}t,\mathbf{y}) - \chi(t,\mathbf{y})$, that is, is understood in the sense of Itô.

Proof. Indeed, by representing G_t as $G_t(\mathbf{y}) = (G(\mathbf{y}) - I)1_t + I = I + 1_t[G - I](\mathbf{y})$, for $\chi(t) = U(t - t_0)G_t(-t_0)\eta$ one obtains the integral equation

$$\chi(t) = U(t - t_0)\eta + U(t)1_t[G - I]U(-t_0)\eta =$$
$$= e^{-\mathrm{i}Ht/\hbar}\left(e^{\mathrm{i}Ht_0/\hbar}\eta + \int_{t_0}^{t} e^{\mathrm{i}Hr/\hbar}\mathrm{d}1_r[G - I]\chi(r)\right).$$

But this equation is equivalent to the differential equation (**??**), a fact that can be proved by straightforward differentiation taking into account the Itô multiplication table

$$(\mathrm{d}t)^2 = 0\,, \quad \mathrm{d}t\mathrm{d}1_t = 0 = \mathrm{d}1_t\mathrm{d}t\,, \quad (\mathrm{d}1_t)^2 = \mathrm{d}1_t\,.$$

■

Similarly, one can obtain the simplest nonlinear stochastic equation for the normalized reduced state vector $\chi_y(t) = \chi(t, \mathbf{y})/\|\chi(t, \mathbf{y})\|$:

$$\mathrm{d}\chi_y(t) + \mathrm{i}/\hbar H\chi_y(t)\mathrm{d}t = (G_y(t) - I)\chi_y(t)\mathrm{d}1_t\,, \quad \chi_y(0) = \eta,$$

where $G_y(t) = G(\mathbf{y})/\|G(\mathbf{y})\chi_y(t)\|$, $\eta \in \mathcal{H}$. This equation is an equivalent differential form of the nonlinear integral stochastic equation

$$\chi_y(t) = e^{-\mathrm{i}Ht/\hbar}\left(e^{\mathrm{i}Ht_0/\hbar}\eta + \int_{t_0}^{t} e^{\mathrm{i}Hr/\hbar}(G_y(r) - I)\chi_y(r)\mathrm{d}1_r\right) =$$
$$= U(t - t_0)\eta + U(t)(G_y - I)1_t\chi_y(0) = U(t)G_t(\mathbf{y})\chi_y(0)/\|G_t(\mathbf{y})\chi_y(0)\|.$$

This yields $\chi_y(t) = T(t, \mathbf{y})\eta/\|T(t, \mathbf{y})\eta\|$ for an $t_0 \leqslant 0$ because of $\|G_t(\mathbf{y})\eta\| = \|T(t, \mathbf{y})\eta\|$.

Note that the random state vector $\chi_y(t)$ is obtained by conditioning with respect to the output (rather than input) probability measure $\mathrm{d}\mu = \|\chi(t, \mathbf{y})\|^2\mathrm{d}\mu_0$.

3 Spontaneous Localization of a Single Particle

Let us consider a spontaneous process of scattering interactions (2.1) of a quantum particle at random time instants $t_n > 0$, $t_1 < t_2 < \ldots$, with a renewable meter in an apparatus of the cloud chamber type with bubbles serving as the meter. We consider the increasing sequences $(t_1, t_2, \ldots)$ as countable subsets $\tau \subset \mathbb{R}_+$ such that $\tau_t = \tau \cap [0, t)$ is finite for any $t \geqslant 0$ in accordance with the finiteness of the number of scattered bubbles on the finite observation interval $[0, t)$. The set of all such infinite τ will be denoted by Γ_∞, Γ is the inductive limit $\cup\Gamma_t$ as $t \to \infty$ of $\Gamma_t = \{\tau_t : \tau \in \Gamma_\infty\}$, which is equal to the disjoint union $\Gamma_t = \sum_{n=0}^{\infty} \Gamma_t(n)$ of n-simplices $\Gamma_t(n) = \{t_1 < \cdots < t_n\} \subset [0, t)^n$.

The measurement apparatus is assumed to be a quantum system of infinitely many bubbles each of which is identical to the single meter described in the previous section. The coordinates q_n^α, $\alpha = 1, \ldots, d$, of a bubble labeled by the scattering number $n \in \mathbb{N}$ show the position $\mathbf{y}_n \in \Lambda$ of the pointer at time $t_n \in \tau$.

The interaction Hamiltonian of the particle corresponding to these scatterings is given by the series

$$H(t,\tau) = H_0 + \kappa \mathbf{R} \otimes \sum_{n=1}^{\infty} \delta(t - t_n) \mathbf{P}(n) \tag{3.1}$$

having at most two nonzero terms when $t \in \tau$. Here $H_0 = H \otimes \mathbf{1}$ is the Hamiltonian describing the time evolution on the intervals between the scatterings $t \in \tau$ and $\mathbf{P}(n)$ is the momentum of the nth scattered bubble, given as the vector-operator $\mathbf{P}(n) = (P_1(n), \dots, P_d(n))$, where $P_\alpha(n) = -\mathrm{i}\hbar \mathrm{d}/\mathrm{d}q_n^\alpha$ in the case $\Lambda = \mathbb{R}^d$, and $\mathbf{R} \otimes \mathbf{P}(n) = \sum P_a(n) R^a$.

The generalized Schrödinger equation corresponding to the Hamiltonian (2.1) can be written for fixed $\tau \in \Gamma_\infty$ by analogy with the single-kick case

$$\mathrm{d}\psi(t) + \frac{\mathrm{i}}{\hbar} H_0 \psi(t)\, \mathrm{d}t = (S(n_t) - I)\psi(t)\, \mathrm{d}n_t, \quad \psi(0,\tau) = \psi_0. \tag{3.2}$$

Here $S(n) = \exp\{-(\mathrm{i}/\hbar)\kappa \mathbf{R} \otimes \mathbf{P}(n)\}$ and $n_t(t) = |\tau_t|$ is the numerical process that gives the cardinality $|\tau_t| = \sum_{r\in\tau} 1_{t-r}$ of the localized subset $\tau_t = \{t_n < t\}$, so that $\mathrm{d}n_t(t)$ is equal to 1 for $t \in \tau$, and zero otherwise.

Proposition 2 *The solution to the equation (3.2) is uniquely determined for every $\tau \in \Gamma_\infty$ by the initial state ψ_0 of the system. Namely, $\psi(t,\tau) = U(t,\tau)\psi_0$, where $U(t,\tau) = U_0(t)V_t^\dagger(\tau)$, $V_t^\dagger(\tau)$ is the chronological product $\prod_{r\in\tau}^{\leftarrow} S_t(r) = S(t_{n_t}) \dots S(t_1)$, and*

$$V_t(\tau) = S_t^\dagger(t_1) S_t^\dagger(t_2) \cdots = \left(\prod_{r\in\tau}^{\leftarrow} S_t(r) \right)^\dagger. \tag{3.3}$$

Here $S_t(t_n) = U_0^\dagger(t_n) S_t(n) U_0(t_n)$ for $t_n < t$, where $S(n) = \exp\{-(\mathrm{i}/\hbar)\, \kappa \mathbf{R} \otimes \mathbf{P}(n)\}$, and $S_t(t_n) = I$ for $t_n \geqslant t$, so that the infinite product (3.3) contains only a finite number $n_t = \sum_{r\in\tau} 1_{t-r}$ of factors different from the identity operator I.

Proof. Recall that the differential equation (3.2) is equivalent to the integral equation given by the recurrence relation

$$\psi(t,\tau) = e^{-\mathrm{i}H_0 t/\hbar} \left(\psi_0 + \sum_{r\in\tau}^{r<t} e^{\mathrm{i}H_0 r/\hbar} (S(n_r) - I)\psi(r,\tau) \right) \tag{3.4}$$

for every $\tau \in \Gamma_\infty$. Hence, $\psi(t,\tau) = U_0(t)V_t^\dagger(\tau)\psi_0$, where $U_0(t) = e^{-\mathrm{i}H_0 t/\hbar}$ and $V_t(\tau)$ is a solution to the operator equation

$$V_t(\tau) = I + \sum_{r\in\tau}^{r<t} V_r(t)(S_t^\dagger(r,\tau) - I)\,, \quad V_0(\tau) = I\,,$$

where $S^{\dagger}(t,\tau) = U_0(t)^{\dagger} S(n_t(\tau))^{\dagger} U_0(t)$. But this equation has a unique solution (3.2), which can be written as the binomial sum

$$[L_t(t_1) + I][L_t(t_2) + I] \cdots = \sum_{\rho \subseteq \tau_t} L(r_1, \tau) \dots L(r_n, \tau)$$

in terms of $\rho = \{r_1, \dots, r_n\}$, $r_1 < \cdots < r_n$, $n \leqslant n_t$, $L_t(r) = S_t^{\dagger}(r) - I$ ($= 0$ if $r > t$) and $L(r,\tau) = S^{\dagger}(r,\tau) - I$. Indeed, this sum contains I as the null product corresponding to $r = 0$ and the sum of the other terms is equal to

$$\begin{aligned} V_t(\tau) - I &= \sum_{r \in \tau}^{r<t} \sum_{\rho \subseteq \tau_r}^{\rho < r} L(r_1, \tau) \dots L(r_m, \tau) L(r, \tau) \\ &= \sum_{r \in \tau}^{r<t} V_r(\tau) L(r,\tau) = \sum_{r \in \tau}^{r<t} V_r(\tau)(S^{\dagger}(r,\tau) - I), \end{aligned}$$

where $m \leqslant n_t - 1$. ■

Note that the differential equation (3.2) depending on $\tau \in \Gamma_\infty$ via $n_t = n_t(\tau)$ is not stochastic as long as we have not fixed a probability distribution for the instants $\tau = (t_1, t_2, \dots)$ of the spontaneous interactions. To obtain a continuous (at least in the mean) dynamics for such an instantaneous process, one can assume that the probability distribution of the random number process $n_t(\tau)$ is given by the Poisson law $\pi_0(\mathrm{d}\tau)$ on Γ_∞ presented as the projective limit as $t \to \infty$ of the probability measures

$$\pi_0(\mathrm{d}\tau_t) = e^{-\nu t} \nu^{|\tau_t|} \, \mathrm{d}\tau_t \,, \quad \nu > 0 \,. \tag{3.5}$$

Here $\tau_t = \tau$ is a finite time-ordered sequence $\tau(n) = (t_1, \dots, t_n) \in \Gamma_t$ with $n = n_t$, $\mathrm{d}\tau_t = \prod_{k=1}^{n_t} \mathrm{d}t_k$ is the measure on Γ_t given by the sum of product measures $\mathrm{d}t_1, \dots, \mathrm{d}t_n = \mathrm{d}\tau(n)$ on the simplices $\Gamma_t(n)$, $\mathrm{d}\tau(0) = 1$ on $\Gamma_t(0) = \{\varnothing\}$ such that

$$\int_{\Gamma_t} \nu^{|\tau|} \mathrm{d}\tau := \sum_{n=0}^{\infty} \nu^n \int \dots \int_{0 \leqslant t_1 < \cdots < t_n < t} \mathrm{d}t_1 \dots \mathrm{d}t_n = e^{\nu t} \,.$$

Note that any other numerical process can be described by a positive density function $f(\tau)$ with respect to the Poissonian measure, that is, has the form $f(\tau)\pi(\mathrm{d}\tau)$.

Let us fix an initial state $\varphi_0 = f_0^{\infty}$ of the apparatus as the infinite product $f_0^{\infty} = \otimes_{k=1}^{\infty} f_k$ of the identical state vectors $f_k = f_0$ of the bubbles given by a normalized element $f_0 \in L^2(\Lambda)$. We suppose, as in §1, that $f_0(\mathbf{y}) \neq 0$ for almost all $\mathbf{y} \neq \Lambda$ such that the space $\mathcal{E} = \{\varphi : \Lambda \to \mathbb{C} : \|\varphi\|_0^2 < \infty\}$ is isomorphic to $L^2(\Lambda)$ with respect to the multiplication $\varphi(\mathbf{y}) = f(\mathbf{y})/f_0(\mathbf{y})$ by f_0^{-1} and the scalar product $\|\varphi\|_0^2 = \int |\varphi(\mathbf{y})|^2 \mathrm{d}\mu_0 = \|f\|^2$. This defines the solutions $\psi(t,\tau)$ of the stochastic equation (3.2) with the initial data $\psi_0 = \eta \otimes \varphi_0$ given by state vectors $\eta \in \mathcal{H}$ in the particle space $\mathcal{H}$ as the state vectors in the product space $\mathcal{H}_\infty = \mathcal{H} \otimes \mathcal{E}_\infty$, where $\mathcal{E}_\infty = \lim_{n \to \infty} L^2(\Lambda^n)$ is the Hilbert space generated by

the infinite-product functions $\varphi(\boldsymbol{y}) \simeq \prod_{k=1}^{\infty} f_k(\mathbf{y}_k)$ with $f_k = f_0$ for almost all k. One can describe $\mathcal{E}_\infty$ as the space of all functions $\varphi : \Lambda^\infty \to \mathbb{C}$ square integrable with respect to the product measure $\mu_0^\infty(\mathrm{d}\boldsymbol{y}) = \mu_0(\mathrm{d}\mathbf{y}_1)\cdot\mu_0(\mathrm{d}\mathbf{y}_2)\cdots$ on the space $\Lambda^\infty = \Lambda\times\Lambda\times\dots$ of sequences $\boldsymbol{y} = (\mathbf{y}_1, \mathbf{y}_2, \dots)$: $\|\varphi\|_0^2 = \int |\varphi(\boldsymbol{y})|^2 \mu_0^\infty(\mathrm{d}\boldsymbol{y}) < \infty$). The generalized product vectors $|\boldsymbol{y}\rangle = |\mathbf{y}_1\rangle \otimes |\mathbf{y}_2\rangle \otimes \dots$ of this space are defined by the δ-functions $\langle \mathbf{q}|\mathbf{y}\rangle$ normalized with respect to $\mu_0 : \int |\boldsymbol{y}\rangle\langle\boldsymbol{y}|\mu_0^\infty(\mathrm{d}\boldsymbol{y}) = 1$.

Consider the sequence $\boldsymbol{q} = (\mathbf{q}_1, \mathbf{q}_2, \dots)$ of coordinates q_n^α, $\alpha = 1, \dots, d$, of the scattered bubbles at the time instants $\{t_1, t_2, \dots\}$. The commuting vector-operators $\mathbf{q}_n$, $n \in \mathbb{N}$, described in $\mathcal{E}_\infty$ by the multiplications $\mathbf{q}_n|\boldsymbol{y}\rangle = |\boldsymbol{y}\rangle\mathbf{y}_n$, are assumed to be measured at the random time instants t_n, $n \in \mathbb{N}$. The point trajectories of such measurements are given by the sequences $\boldsymbol{y} = (y_1, y_2, \dots)$ of pairs $y_n = (t_n, \mathbf{y}_n)$ with $t_1 < t_2, \dots$ and $\mathbf{y}_n \in \Lambda$, identified with countable subsets $\upsilon = \{y_1, y_2, \dots\} \subset \mathbb{R}_+ \times \Lambda$. As elements $\upsilon = (\tau, \boldsymbol{y})$ of the Cartesian product $\Upsilon_\infty = \Gamma_\infty \times \Lambda^\infty$, they have the probability distribution $\mathrm{P}_0(\mathrm{d}\upsilon) = \pi_0(\mathrm{d}\tau)\mu_0^\infty(\mathrm{d}\boldsymbol{y})$, where Λ^∞ is the space of all sequences $\boldsymbol{y} = (\mathbf{y}_1, \mathbf{y}_2, \dots)$, $\mathbf{y}_n \in \Lambda$, equipped with the probability product-measure $\mu_0^\infty(\mathrm{d}\boldsymbol{y})$.

The measurement data of the observable process up to a given time instant $t > 0$ is described by a finite sequence $\upsilon_t = (y_1, \dots, y_n)$ with $n = n_t(\upsilon)$ given by the numerical process $n_t(\tau)$ for the component τ of υ.

Let us introduce the counting distribution $n_t(\Delta) = |\upsilon_t \cap (\mathbb{R}_+ \times \Delta)|$ as the number $n_t(\Delta, \upsilon)$ of scatterings in the time-space region $[0, t) \times \Delta$ and define the counting integral $\int_0^\infty \int_\Lambda L(r, \mathbf{y})\mathrm{d}n_r(\mathrm{d}\mathbf{y})$ over $y \in \mathbb{R}_+ \times \Lambda$ as the series

$$n[L](\upsilon) = \sum_{y\in\upsilon} L(y)\,, \quad \forall\, \upsilon \in \Upsilon_\infty\,. \tag{3.6}$$

Having fixed an integer-valued distribution $n_t(\Delta) \in \{0, 1, \dots\}$ as a function of $t \geqslant 0$ and of measurable sets $\Delta \subseteq \Lambda$, one can obtain the corresponding trajectory υ as a sequence of the counts of the jumps of $n_t(\Delta)$ in the time-space $\mathbb{R}_+ \times \Lambda$.

Given an initial state vector in $\mathcal{H}_\infty$ of the form $\psi_0 = \eta \otimes \varphi_0$ with fixed $\varphi_0 \simeq f_0^\infty$, one can define a nonunitary stochastic evolution $\eta \mapsto T(t, \upsilon)\eta$ by setting

$$T(t, \upsilon) = \langle \boldsymbol{y}|U(t, \tau)\varphi_0\,, \quad \upsilon = (\tau, \boldsymbol{y}),$$

which reproduces the unitary evolution $U(t, \tau) = U_0(t)V_t^\dagger(\tau)$ defined by (3.2). This can also be written as $T(t, \upsilon) = U(t)F_t^\dagger(\upsilon)$, since $U_0(t) = U(t) \otimes \mathbf{1}$ commutes with the (generalized) eigen-bras $\langle\boldsymbol{y}| = \langle \mathbf{y}_1, \mathbf{y}_2, \dots|$ of the bubble coordinates $(\mathbf{q}_1, \mathbf{q}_2, \dots) : \langle\boldsymbol{y}|U_0(t) = U(t)\langle\boldsymbol{y}|$. The reduction transformations $F_t(\upsilon)$, $\upsilon \in \Upsilon_\infty$, are given by the chronological products

$$F_t(\upsilon) = G_t^\dagger(y_1)G_t^\dagger(y_2)\cdots \equiv \overrightarrow{\prod_{y\in\upsilon}} G_t^\dagger(y) \tag{3.7}$$

of $G_t(t_n, \mathbf{y}_n) = U^\dagger(t_n)G(\mathbf{y}_n)U(t_n)$ for $t_n < t$, where $G(\mathbf{y}) = \langle \mathbf{y}|Sf_0$, owing to the product form (3.3) of the unitary transformations $V_t(\tau)$, $\tau \in \Gamma$, and $\langle\boldsymbol{y}| = \otimes_{k=1}^\infty \langle \mathbf{y}_k|$, $\varphi_0(\boldsymbol{y}) \simeq \prod_{k=1}^\infty f_0(\mathbf{y}_k)$, $\langle \boldsymbol{y}|\varphi_0 \equiv 1$ for $\boldsymbol{y} \in \Lambda^\infty$. The stochastic

operator (3.7) defined by the single-point reductions

$$G_t(r,\mathbf{y}) = \langle \mathbf{y}|S_t(r)f_0 = \begin{cases} G(r,\mathbf{y})\,, & r \leqslant t \\ I, & r > t \end{cases} \tag{3.8}$$

is normalized with respect to the initial probability $\mathrm{P}_0(\mathrm{d}\upsilon)$, $\Pi_{\Upsilon_\infty}[I](t) = I$, where

$$\Pi_A[X](t) = \int_A T^\dagger(t,\upsilon)XT(t,\upsilon)\mathrm{P}_0(\mathrm{d}\upsilon) = \int_A F_t(\upsilon)XF_t^\dagger(\upsilon)\mathrm{P}_0(\mathrm{d}\upsilon)\,, \tag{3.9}$$

is a continual operational-valued measure [3]–[5] defined on measurable sets $A \subseteq \Upsilon_\infty$ of the point trajectories $\upsilon_t = \{(r,\mathbf{y}) \in \upsilon | r < t\}$ given by the operations $\Phi_t(\upsilon) : X \mapsto F_t(\upsilon)XF_t^\dagger(\upsilon)$ for particle operators $X : \mathcal{H} \to \mathcal{H}$.

The positive operator-valued measure $\Pi_t(A) = \Pi_A[I](t)$ gives the statistics

$$\mathrm{P}_t(A) = \int_A \|(\boldsymbol{y}|U(t,\tau)\psi_0\|^2 \mu_0^\otimes(\mathrm{d}\boldsymbol{y})\pi_t(\mathrm{d}\tau)$$

for the continual observation with respect to an arbitrary initial wave-function $\psi_0 = \eta \otimes \varphi_0$, $\eta \in \mathcal{H}$ in the form

$$\mathrm{P}_t(A) = \langle \eta, \Pi_t(A)\eta \rangle\,.$$

The output probability measure $\mathrm{P}(A)$, $A \subseteq \Upsilon_\infty$, is defined by the marginales $\mathrm{P}_t(A)$, $A \subseteq \Upsilon_t$, as $t \to \infty$.

Theorem 2 *The reduced wave function $\chi(t,\upsilon) = T(t,\upsilon)\eta$ is normalized*

$$\|\chi(t)\|^2 = \int \|\chi(t,\upsilon)\|^2 \mathrm{P}_0(\mathrm{d}\upsilon) = 1$$

as a stochastic vector process $\chi(t) : \Upsilon_\infty \to \mathcal{H}$ with respect to the initial probability P_0. It satisfies the stochastic wave equation

$$\mathrm{d}\chi(t) + \frac{\mathrm{i}}{\hbar} H\chi(t)\,\mathrm{d}t = \mathrm{d}n_t[G-I]\chi(t)\,, \quad \chi(0) = \eta, \tag{3.10}$$

expressed in terms of the random differential $\mathrm{d}n_t[G-I](\upsilon) = (G(\mathbf{y}_{n_t}(\upsilon)) - 1)\mathrm{d}n_t(\upsilon)$, $n_t(\upsilon) = n_t(\tau)$ for the point distribution $n_t[L] = \int_\Lambda L(\mathbf{y})n_t(\mathrm{d}\mathbf{y}) = n[L_t]$ over $\mathbf{y} \in \Lambda$ defined as (3.6) with $L_t(r,\mathbf{y}) = 1_t(r)L(\mathbf{y})$.

Proof. To prove Eq. (3.10), discovered for the first time in [13], we rewrite it in the following integral form

$$\chi(t) = e^{-\mathrm{i}Ht/\hbar}\left(\eta + \int_0^t \int_\Lambda \mathrm{e}^{\mathrm{i}Hr/\hbar}(G(\mathbf{y}) - I)\chi(r)\mathrm{d}n_r(\mathrm{d}\mathbf{y})\right),$$

given for each $\upsilon \in \Upsilon_\infty$ by the finite sum

$$n[1_t U^\dagger(r)(G-I)\chi](\upsilon) = \sum_{(r,\mathbf{y})\in\upsilon}^{r<t} U^\dagger(r)(G(\mathbf{y}) - I)\chi(r)\,.$$

We express the solution to this equation in the form $\chi(t,\upsilon) = U(t)F_t^\dagger(\upsilon)\eta$ via the solution (3.7) to the recursion equation

$$F_t(\upsilon) = I + \sum_{(r,\mathbf{y})\in\upsilon}^{r<t} F_r(\upsilon)(G(r,\mathbf{y}) - I)\,, \quad F_0(\upsilon) = I,$$

with $G(t,\mathbf{y}) = U^\dagger(t)G(\mathbf{y})U(t)$, as was done for the unitary case. ■

Let us also write on F the nonlinear equation

$$\mathrm{d}\chi_\upsilon(t) + \frac{\mathrm{i}}{\hbar}H\chi_\upsilon(t)\mathrm{d}t = (G_\upsilon(t) - I)\chi_\upsilon(t)\mathrm{d}n_t(\upsilon)\,, \quad \chi_\upsilon(0) = \eta,$$

with $G_\upsilon(t) = G(\mathbf{y}_{n_t(\upsilon)})/\|G(\mathbf{y}_{n_t(\upsilon)})\chi_\upsilon(t)\|$. Its solutions define the normalized reduction $\chi_\upsilon(t) = \chi(t,\upsilon)/\|\chi(t,\upsilon)\|$ for continual counting measurements as a stochastic vector process $\chi_\upsilon(t) \in \mathcal{H}$ with respect to the output probability measure P of the point process $t \mapsto \upsilon_t$. This can easily be obtained, as in [14], by applying the Itô multiplication table

$$(\mathrm{d}t)^2 = 0\,, \quad \mathrm{d}t\,\mathrm{d}n_t = 0 = \mathrm{d}n_t\mathrm{d}t\,, \quad (\mathrm{d}n_t)^2 = \mathrm{d}n_t\,.$$

4 Mixing Reduction for Many Particles

We now consider M identical particles interacting independently with the bubbles in accordance with the scattering term in the Hamiltonian (3.1). The spontaneous process of scatterings is described by the time-ordered sequences of pairs (k_n, t_n), $t_1 < t_2 < \dots$, where $k_n \in \{1,\dots,M\}$ is the number of the particle labeled by the scattering number $n \in \mathbb{N}$ at time instant $t_n > 0$. We have excluded the possibility of two or more scatterings of the bubbles at the same instant of time, as was done for a single particle in §2. The sequence (k_1,t_1), $(k_2,t_2),\dots$ of the scatterings can be represented by the occupational subsets $\tau_k = \{t_n \in \tau : k_n = k\}$ of the time set $\tau = \{t_1,t_2,\dots\}$, which are disjoint, $\tau_k \cap \tau_l = \varnothing$ if $k \neq l$, since the scatterings for different particles are independent. We shall consider the M-tuples $\tau_\bullet = (\tau_1,\dots,\tau_M)$ of these countable subsets $\tau_k \subset \mathbb{R}_+$ as elements $\tau_\bullet \in \Gamma_\infty^M$ of the Cartesian M-product Γ_∞, given by the partition $\tau = \sqcup\tau_k := \cup\tau_k$, $\tau_k \cap \tau_l = \varnothing$ if $k \neq l$ of a $\tau \in \Gamma_\infty$.

The interaction Hamiltonian of M particles with independent scatterings labled by a sequence $\tau_\bullet = \{(k_1,t_1),\,(k_2,t_2),\dots\}$ reads

$$H(t,\tau_\bullet) = H_0^M + \kappa\sum_{n=1}^{\infty}\mathbf{R}(k_n))\otimes\delta(t-t_n)\mathbf{P}(n). \tag{4.1}$$

Here $H_0^M = H^M \otimes \mathbf{1}$ is the Hamiltonian of the particles describing the time evolution on the intervals between the scatterings with the bubbles:

$$H^M = \sum_{k=1}^{M} H(k) + \sum_{k=1}^{M}\sum_{l>k}^{M} W(k,l)\,,$$

where $H(k) = I^{\otimes(k-1)} \otimes H \otimes I^{\otimes(M-k)}$ is the Hamiltonian of the kth particle and $W(k,l)$ is the interaction potential in $\mathcal{H}^{\otimes M}$ of the kth and lth particle, $1 \leqslant k < l \leqslant M$.

Let $\mathcal{H}^M$ denote the M-particle Hilbert space, which is an invariant subspace $\mathcal{H}^M \subseteq \mathcal{H}^{\otimes M}$ of symmetric (bosons) or antisymmetric (fermions) M-tensors $\eta^M \in \mathcal{H}^{\otimes M}$ generated by the product-vectors $\otimes_{k=1}^M \eta_k \in \mathcal{H}^{\otimes M}$ with $\eta_k \in \mathcal{H}$. The correspondent Itô-Schrödinger equation for the stochastic state vector $\psi^M(t) : \tau_\bullet \mapsto \psi(t,\tau_\bullet)$ with values $\psi(t,\tau_\bullet) \in \mathcal{H}_\infty^M$ in the product space $\mathcal{H}_\infty^M = \mathcal{H}^M \otimes \mathcal{E}_\infty$ of the M-particle space $\mathcal{H}^M$ by $\mathcal{E}_\infty = \lim_{n\to\infty} L^2(\Lambda^n)$ reads

$$\mathrm{d}\psi^M(t) + \frac{\mathrm{i}}{\hbar} H_0^M \psi^M(t)\mathrm{d}t = (S(k_t, n_t) - I)\psi^M(t)\mathrm{d}n_t \,. \tag{4.2}$$

Here $S(k,n) = \exp\{-\frac{\mathrm{i}}{\hbar}\kappa \mathbf{R}(k) \otimes \mathbf{P}(n)\}$, $\mathbf{R}(k) = I^{\otimes(k-1)} \otimes \mathbf{R} \otimes I^{\otimes(M-k)}$,

$$k_t(\tau_\bullet) = \sum_{k=1}^M k 1_{\tau_k}(t)\,, \quad \text{where } 1_\tau(t) = \begin{cases} 1\,, & t \in \tau \\ 0\,, & t \notin \tau \end{cases}$$

is the random number $k_t : \Gamma_\infty^M \to \{1, \dots, M\}$, labeling a particle by k at any instant $t \in \tau_k$ of its collision with a bubble labeled by $n_t(\tau) = \sum_{k=1}^M n_{k,t} = |\tau \cap (0,t)|$, where $n_{k,t} = |\tau_k \cap [0,t)|$, $\tau = \cup \tau_k$.

Proposition 3 *The solutions $\psi(t,\tau_\bullet) = U(t,\tau_\bullet)\psi_0^M$, $\psi_0^M \in \mathcal{H}_\infty^M$, of Eq. (4.2) can be written as $U(t,\tau_\bullet) = U_0^M(t) V_t^\dagger(\tau_\bullet)$ in terms of the finite chronological product*

$$V_t(\tau_\bullet) = S_t^\dagger(k_1, t_1) S_t^\dagger(k_2, t_2) \dots, \quad \tau_\bullet \in \Gamma_\infty^M \,, \tag{4.3}$$

where $S_t(k, t_n) = U_0^{M\dagger}(t_n) S(k,n) U_0^M(t_n)$ and $U_0^M(t) = \exp\{-\frac{\mathrm{i}}{\hbar} H_0^M(t)\}$.

The proof is exactly the same as for the case of a single particle ($M = 1$).

Let $\omega = (w_1, w_2, \dots)$ denote a chronologically ordered sequence of triples $w_n = (k_n, t_n, \mathbf{y}_n)$ and Ω the space of such sequences with $\{t_1, t_2, \dots\} \in \Gamma_\infty$. Every sequence $\omega \in \Omega$ can be represented as a pair $\omega = (\tau_\bullet, \boldsymbol{y})$, where $\tau_\bullet = (\tau_1, \dots, \tau_M)$ is a partition of the corresponding sequence $\tau = \{t_1, t_2, \dots\}$ and $\boldsymbol{y} = (\mathbf{y}_1, \mathbf{y}_2, \dots)$, so that Ω can be identified with the product $\Gamma_\infty^M \times \Lambda^\infty$. The space Ω is equipped with the probability measure $\mathrm{P}_0(\mathrm{d}\omega) = \pi_0(\mathrm{d}\tau_\bullet)\mu_0^\infty(\mathrm{d}\boldsymbol{y})$, where $\pi_0(\mathrm{d}\tau_1, \dots, \mathrm{d}\tau_M) = \prod_{k=1}^M \pi_0(\mathrm{d}\tau_k)$ is the product of the identical Poisson measures (3.5), in accordance with the independence of spontaneous interactions of each particle with the bubbles.

Given an initial state vector $\psi^M = \eta^M \otimes \varphi_0$, where $\varphi_0 \simeq f_0^\infty$, one can easily prove that the nonunitary stochastic evolution

$$T(t,\omega) = \langle \boldsymbol{y} | U(t,\tau_\bullet)\varphi_0 \,, \quad \omega = (\tau_\bullet, \boldsymbol{y})$$

is also a finite chronological product

$$T(t,\omega) = U^M(t) F_t^\dagger(\omega)\,, \quad F_t(\omega) := G_t^\dagger(w_1) G_t^\dagger(w_2) \dots \,.$$

Here $G_t(k, t_n, \mathbf{y}) := I$ for $t_n > t$ and

$$G_t(k, t_n, \mathbf{y}) := U^{M\dagger}(t_n) G(k, \mathbf{y}) U^M(t_n), \ t_n \leqslant t,$$

where $U^M(t) = \exp\{-\frac{\mathrm{i}}{\hbar} H^M t\}$, is defined by the reduced scattering operator

$$G(k, \mathbf{y}) = I^{\otimes(k-1)} \otimes G(\mathbf{y}) \otimes I^{\otimes(M-k)}, \quad G(\mathbf{y}) = \langle \mathbf{y}| S f_0 \tag{4.4}$$

for $f_0 \in L^2(\Lambda)$, $\|f_0\| = 1$, applied to the kth particle only in $\mathcal{H}^M$. The stochastic operator $T(t)$ defines the solutions $\chi^M(t, \omega) = T(t, \omega)\eta^M$ to the Itô differential equation

$$\mathrm{d}\chi^M(t) + \frac{\mathrm{i}}{\hbar} H^M \chi^M(t)\, \mathrm{d}t = \sum_{k=1}^{M} \mathrm{d}n_{k,t}[G(k) - I]\chi^M(t), \quad \chi^M(0) = \eta^M$$

for the stochastic vector states $\chi^M(t) : \Omega \to \mathcal{H}^M$ of the M-particle system, corresponding to an initial $\eta^M \in \mathcal{H}^M$. The right-hand side of this equation is written as a point integral $n_t[L] = \sum_{k=1}^{M} \int L(k, \mathbf{y})\, \mathrm{d}n_t = n[L_t]$ with respect to the stochastic distribution $n[L](\omega) = \sum_{w \in \omega} L(w)$ for $L(k, r, \mathbf{y}) = 1_t(r) L(k, \mathbf{y}) \equiv L_t(w)$. The vector $\chi^M(t, \omega) \in \mathcal{H}^M$ as well as $\chi(t, \upsilon)$ in (3.10), is no longer normalized ($\|\chi^M(t, \omega)\| \neq 1$) for an initial state-vector $\eta^M \in \mathcal{H}^M$, $\|\eta^M\| = 1$, but it is normalized with respect to the probability measure P_0 on Ω in the mean square sense. But, in contrast to $\chi(t, \upsilon)$, the vector $\chi^M(t, \omega)$ is not yet the reduced description of the M-particle system under the observation of the scattering process $\upsilon_t = \{(r, \mathbf{y}) \in \upsilon : r \leqslant t\}$, given by the registration of the pointer positions $\mathbf{y}_n \in \Lambda$ at random time instants t_n.

The reduced dynamics corresponding to the observation is described by a stochastic operational process $X \mapsto \Theta[X](t)$,

$$\Theta[X](t, \upsilon) = \Phi_t[U^{M\dagger}(t) X U^M(t)](\upsilon) \tag{4.5}$$

for the M-particle operators $X : \mathcal{H}^M \to \mathcal{H}^M$ given by the conditional expectation

$$\Phi_t[X](\tau, \boldsymbol{y}) = \frac{1}{M^{|\tau_t|}} \sum_{\sqcup \sigma_k = \tau_t} F_t(\sigma_\bullet, \boldsymbol{y}) X F_t^\dagger(\sigma_\bullet, \boldsymbol{y}) \tag{4.6}$$

where the sum is taken over all partitions $\sigma_\bullet = (\sigma_1, \ldots, \sigma_M)$ of a finite subset $\tau_t = \tau \cap [0, t)$. This averaging is due to the impossibility to detect the individuality of the identical particles producing indistinguishable effects on the bubbles by measuring the scatterings of the bubbles.

To prove Eq. (4.6), we need to compare the correlations of $F_t(\omega) X F_t^\dagger(\omega)$ and of an arbitrary functional $g(\upsilon_t)$ of the observable point process υ_t with the correlations of (4.6) and of $g(\upsilon_t)$. But by applying the well-known formula [21] one can easily find

$$\int_{\Gamma_t^M} x(\sigma_1, \ldots, \sigma_M) \prod_{k=1}^{M} \mathrm{d}\sigma_k = \int_{\Gamma_t} \sum_{\sigma_k : \sqcup \sigma_k = \sigma} x(\sigma_1, \ldots, \sigma_M)\, \mathrm{d}\sigma$$

for the multiple point integration that these correlations with respect to the probability measure P_0 on Ω given by the Poisson law (3.5) simply coincide:

$$\int_{\Gamma_t^M} \pi_t(\mathrm{d}\boldsymbol{\sigma}_\bullet) \int_{\Lambda^\infty} g(\sqcup_{k=1}^M \sigma_k, \boldsymbol{y}) F(\boldsymbol{\sigma}_\bullet, \boldsymbol{y}) X F_t^\dagger(\boldsymbol{\sigma}_\bullet, \boldsymbol{y}) \mu_0^\infty(\mathrm{d}\boldsymbol{y}) =$$
$$= \int_{\Gamma_t^M} \langle g(\sqcup_{k=1}^M \sigma_k), X(\sigma_1, \ldots, \sigma_M)\rangle_0 \prod_{k=1}^M e^{-\nu t} \nu^{|\sigma_k|} \mathrm{d}\sigma_k$$
$$= \int_{\Gamma_t} \langle g(\sigma), \sum_{\sigma_k : \sqcup \sigma_k = \sigma} X(\sigma_1, \ldots, \sigma_M)\rangle_0 e^{-M\nu t} \nu^{|\sigma|} \mathrm{d}\sigma$$
$$= \int_{\Gamma_t} \pi_t^M(\mathrm{d}\sigma) \int_{\Lambda^\infty} g(\sigma, \boldsymbol{y}) \frac{1}{M^{|\sigma|}} \sum_{\tau_k : \cup \tau_k = \sigma} F(\omega) X F^\dagger(\omega) \mu_0^\infty(\mathrm{d}\boldsymbol{y}) \,.$$

Here $\langle \cdot, \cdot \rangle_0$ is the abbreviation for the inner product in $\mathcal{E}$ of the test function $\upsilon \mapsto g(\tau, \boldsymbol{y})$ with fixed $\tau \in \Gamma_t$ and the operator function $X_t(\boldsymbol{\tau}_\bullet, \boldsymbol{y}) = F_t(\boldsymbol{\tau}_\bullet, \boldsymbol{y}) X F_t^\dagger(\boldsymbol{\tau}_\bullet, \boldsymbol{y})$ with fixed $\boldsymbol{\tau}_\bullet \in \Gamma_t^M$ and $\tau = \sqcup_{k=1}^M \tau_k$. The probability measure

$$\pi_0^M(\mathrm{d}\tau_t) = \sum_{\sqcup \sigma_k = \tau_t} \prod_{k=1}^M \pi_0(\mathrm{d}\sigma_k) = e^{-M\nu_t} |M\nu|^{|\tau_t|} \mathrm{d}\tau_t \tag{4.7}$$

on Γ_t has the intensity $M\nu$. It is induced by the measure $\pi_0(\mathrm{d}\boldsymbol{\tau}_\bullet)$ on Γ_∞^M with respect to the particle identification map $\boldsymbol{\tau}_\bullet \in \Gamma_\infty^M \mapsto \tau = \cup_{k=1} \tau_k$, defining the observable data $\upsilon = (\tau, \boldsymbol{y})$ by the stochastic map $\omega = (\boldsymbol{\tau}_\bullet, \boldsymbol{y}) \mapsto (\tau = \sqcup_{k=1}^M \tau_k, \boldsymbol{y}) \in \Upsilon_\infty$ on $\omega \in \Omega$.

By the coincidence of the correlations proved above, the stochastic operator (4.5) is indeed the conditional expectation of the stochastic operators $X_t(\omega)$ with respect to the observable process $\tau = \upsilon_t$.

In contrast to the pure operations $X \mapsto X_t(\omega)$, the reduction operation $X \mapsto \Phi_t[X](\upsilon)$ preserves the symmetry of the M-particle operators X with respect to particle permutation. It is the least mixing operation which preserves the indistinguishability of the particles with respect to the observations of the bubble scatterings, corresponding to the complete nondemolition measurement of the particles.

Indeed, the reduction operation (4.6) can be simply written as the finite iteration

$$\frac{1}{M^n} \sum_{k_1, \ldots, k_n = 1}^M G_t^\dagger(k_1, y_1) \cdots G_t^\dagger(k_n, y_n) X G_t(k_n, y_n) \cdots G_t(k_1, y_1) =$$
$$= \Psi_t[\ldots \Psi_t[X](y_1) \ldots](y_n) = \Phi_t[X](y_1, \ldots, y_n)$$

with $n = |\tau_t|$ single mixing reductions

$$\Psi[X](y) = \frac{1}{M} \sum_{k=1}^M G^\dagger(k, y) X G(k, y) \,. \tag{4.8}$$

Given as the arithmetic mean value of the permutations for the pure operations $X \mapsto G_t^\dagger(k,y)XG_t(k,y)$, corresponding to the identical operators (4.4), the reductions $X \mapsto \Psi_t[X](y)$ are permutationally symmetric and are not mixing only if the pure operations do not break this symmetry.

The mixing property of the reduced stochastic dynamics $t \mapsto \rho[X](t,\upsilon)$ derived above for the corresponding statistical states

$$\rho^M[X](t,\upsilon) = \langle \eta^M, \Theta[X](t,\upsilon)\eta^M\rangle = \mathrm{Tr}\{X\rho^M(t,\upsilon)\} \tag{4.9}$$

gives an increase in the entropy

$$\sigma^M(t,\upsilon) = -\,\mathrm{Tr}\{\rho^M(t,\upsilon)\ln\rho^M(t,\upsilon)\}$$

for an ensemble of identical particles even under the condition of complete nondemolition observation. According to (4.9), the reduced density operators $\rho^M(t,\upsilon)$ for the system of M identical particles gives the probability density

$$\mathrm{p}^M(t,\upsilon_t) = \mathrm{Tr}\{\rho^M(t,\upsilon_t)\} = \mathrm{P}^M(\mathrm{d}\upsilon_t)/\mathrm{P}_0^M(\mathrm{d}\upsilon_t)$$

of the output process υ_t. Here $\mathrm{P}_0^M = \pi^M \otimes \mu_0^\infty$ is the probability measure on $\Upsilon_\infty = \Gamma_\infty \times \Lambda^\infty$ defined by the Poisson measure (4.7). This means that the a posteriori density operator $\rho^M(t)$ is defined as a stochastic positive trace class operator normalized in the mean sense

$$\|\rho^M(t)\|_1 = \int \mathrm{Tr}\{\rho^M(t,\upsilon)\}\mathrm{P}_0^M(\mathrm{d}\upsilon) = 1\,.$$

Theorem 3 *The density $\rho^M(t)$ satisfies the stochastic operator equation*

$$\mathrm{d}\rho^M(t) + \frac{\mathrm{i}}{\hbar}[H,\rho^M(t)]\mathrm{d}t = \mathrm{d}n_t\Bigg(\frac{1}{M}\sum_{k=1}^M G(k)\rho^M(t)G^\dagger(k) - \rho^M(t)\Bigg), \tag{4.10}$$

which has a unique solution for every initial condition $\rho^M(0,\upsilon) = \rho_0^M$ given by the density operator ρ_0^M for the M-particle states $\rho_0^M[X] = \mathrm{Tr}\{X\rho_0^M\}$.

Proof. Let us prove Eq. (4.10) written in the equivalent integral form

$$\rho^M[X](t) = \rho^M[X(t)](t) + \int_0^t\int_\Lambda \rho^M[\Psi[X(t-r)](\mathbf{y}) - X(t-r)](r)\mathrm{d}n_t(\mathrm{d}\mathbf{y})$$

for an $\rho_0^M[X] = \langle\eta^M, X\eta^M\rangle$, $\eta^M \in \mathcal{H}^M$, where $X(t) = U^M(t)^\dagger X U^M(t)$,

$$U^M(t) = e^{-\mathrm{i}H^M t/\hbar}\,, \quad \Psi[X](\mathbf{y}) = \frac{1}{M}\sum_{k=1}^M G^\dagger(k,\mathbf{y})XG(k,\mathbf{y})\,.$$

Taking into account the fact that the stochastic integral (3.6) in this equation is simply a finite sum for every $\upsilon \in \Upsilon_\infty$ and t, one can write it as a recursive operator equation

$$\Phi_t[X](\upsilon) = X + \sum_{(r,y)\in\upsilon}^{r<t} \Phi_r[\Psi[X](r,\mathbf{y}) - X](\upsilon)\,,$$

for a stochastic operation Φ_t, defining the solutions to Eq. (4.10), as in (4.9), in terms of the composition (4.5).

But such recursive equation has a unique solution $\Phi_t(\upsilon) = \Psi_t(y_1) \circ \Psi_t(y_2) \circ \ldots$, defined for a $\upsilon = (\tau, \boldsymbol{y}) \in \Upsilon_\infty$ as the chronological composition of the maps $\Psi_t(r, \mathbf{y}) : X \mapsto \Psi[X](r, \mathbf{y})$ if $r \leqslant t$ and $\Psi_t[X](r, \mathbf{y}) = X$ if $r > t$. This solution can be found by the iterations

$$\Phi_t(\upsilon) - I = \sum_{(r,\mathbf{y})\in\upsilon}^{r<t} \Phi_r(\upsilon) \circ \Lambda(r, \mathbf{y}) = \sum_{(r,\mathbf{y})\in\upsilon_t} \left(\Lambda(r, \mathbf{y}) + \sum_{(s,\mathbf{y})\in\upsilon_r} \Phi_s(\upsilon) \circ \Lambda(s, \mathbf{y}) \right) \ldots,$$

where $\Lambda(y) = \Psi(y) - \mathrm{I}$, I is the identical map $X \mapsto X$, and by the binomial formula $\Psi_t(y_1) \circ \Psi_t(y_2) \circ \cdots = \sum_{\sigma\subseteq\upsilon_t} \Lambda(z_1) \circ \cdots \circ \Lambda(z_n)$ in terms of $\sigma = \{z_1, \ldots, z_n\}$, $z = (r, \mathbf{y})$, $r_1 < \cdots < r_n$, $n \leqslant n_t$. ■

Let us also write the nonlinear stochastic equation

$$\mathrm{d}\rho_\upsilon^M(t) + \frac{\mathrm{i}}{\hbar}\left[H, \rho_\upsilon^M(t)\right] \mathrm{d}t = \rho_\upsilon^M(t) \circ (\Psi_\upsilon(t) - \mathrm{I})\, \mathrm{d}n_t(\upsilon)\,,$$

where $\Psi_\upsilon(t) = \Psi(\mathbf{y}_{n_t(\upsilon)}) / \operatorname{Tr}\{E(\mathbf{y}_{n_t(\upsilon)})\rho\}$,

$$\rho \circ \Psi(\mathbf{y}) = \frac{1}{M} \sum_{k=1}^{M} G(k, \mathbf{y}) \rho G^\dagger(k, \mathbf{y})\,, \quad E(\mathbf{y}) = \frac{1}{M} \sum_{k=1}^{M} G^\dagger(k, \mathbf{y}) G(k, \mathbf{y}))$$

for the normalized density operator $\rho_\upsilon^M(t) = \rho^M(t, \upsilon)/\mathrm{p}^M(t, \upsilon)$. This describes the conditional expectations $\rho_\upsilon^M[X](t) = \operatorname{Tr}\{X\rho_\upsilon^M(t)\}$ of the M-particle operators with respect to the output probability measure $\mathrm{P}^M(\mathrm{d}\upsilon) = \mathrm{p}^M(t, \upsilon)\mathrm{P}_0^M(\mathrm{d}\upsilon)$, where $\mathrm{p}^M(t, \upsilon) = \operatorname{Tr}\{\rho^M(t, \upsilon)\}$.

5 Macroscopic and Central Limits of the Model

We now consider the mean field approximation of the measurement apparatus fixing its total effect $\nu\kappa = -\gamma$ given by the mean number ν of scattered bubbles per second and an interaction constant κ coupling each bubble to a particle in the Hamiltonian (3.1). We look for the limits of the unitary and reduced evolutions (3.2) and (3.10) as $\nu \to \infty$ and $\kappa \to 0$ such that γ is a real constant. To perform these limit passages, we need the expansions

$$\begin{aligned} S(n) &= I \otimes \mathbf{1} - \mathrm{i}\frac{\kappa}{\hbar} \mathbf{R} \otimes \mathbf{P}(n) - \left(\frac{1}{2}\right)\left(\frac{\kappa}{\hbar}\right)^2 (\mathbf{R} \otimes \mathbf{P}(n))^2 + \ldots \\ G(\mathbf{y}) &= I - \kappa \frac{f_0'(\mathbf{y})}{f_0(\mathbf{y})} \mathbf{R} + \frac{1}{2}\kappa^2 \mathbf{R} \frac{f_0''(\mathbf{y})}{f_0(\mathbf{y})} \mathbf{R} + \ldots \end{aligned} \tag{5.1}$$

of the scattering operator $S(n) = \exp\{-\frac{\mathrm{i}}{\hbar}\kappa \mathbf{R} \otimes \mathbf{P}(n)\}$ and the reduced operator $G(\mathbf{y}) = f_0(\mathbf{y}I - \kappa\mathbf{R})/f_0(\mathbf{y})$ with respect to the coupling constant κ. The first term of the expansion for $S(n)$ disappears in the right-hand side in Eq. (3.2),

whereas the second and third terms give rise to the differentials of the operator-valued stochastic integrals

$$\hat{n}_t[\mathbf{P}] = \int_0^t \mathbf{P}(n_r)\,\mathrm{d}n_r\,, \quad \hat{n}_t[\mathbf{P}^2] = \int_0^t \mathbf{P}(n_r)^2\,\mathrm{d}n_r\,.$$

The corresponding terms

$$n_t\left(\frac{f_0'}{f_0}\right) = \int_0^t \int_\Lambda \frac{f_0'(\mathbf{y})}{f_0(\mathbf{y})}\,\mathrm{d}n_r(\mathrm{d}\mathbf{y})\,, \quad n_t\left(\frac{f''}{f_0}\right) = \int_0^t \int_\Lambda \frac{f_0''(\mathbf{y})}{f_0(\mathbf{y})}\,\mathrm{d}n_r(\mathrm{d}\mathbf{y})$$

on the right-hand side in Eq. (3.10) can also be written as the integrals $\hat{n}_t[L] = \int_0^t L(n_r)\,\mathrm{d}n_r$ with values in operator functions of $\tau \in \Gamma_\infty$

$$\hat{n}_t[L](\tau) = \sum_{n=1}^{n_t(\tau)} L(n)\,, \quad L(n) = \mathbf{1}^{\otimes(n-1)} \otimes L \otimes \mathbf{1}^{\otimes\infty}\,, \tag{5.2}$$

where L is one of the multiplication operators

$$L' = f_0'(\mathbf{q})/f_0(\mathbf{q})\,, \quad L'' = f_0''(\mathbf{q})/f_0(\mathbf{q})\,.$$

Here $f_0'(\mathbf{q})$ denotes the vector of the derivatives $\partial_\alpha f_0(\mathbf{q}) = \partial f_0(\mathbf{q})/\partial q^\alpha$ and $f''(\mathbf{q})$ denotes the matrix with elements $\partial_\alpha\partial_\beta f(\mathbf{q})$. The operator $\hat{n}_t[L](\tau)$ corresponding to a function $L = l(\mathbf{q})$ of the bubble coordinate vector-operator $\mathbf{q}$ in $L^2(\Lambda)$ acts in $\mathcal{E}_\infty$ as the multiplication operator

$$[\hat{n}_t[L](\tau)\varphi](\boldsymbol{y}) = \sum_{n=1}^{n_t(\tau)} l(\mathbf{y}_n)\varphi(\boldsymbol{y}) \equiv n_t[l](\tau, \boldsymbol{\upsilon})\varphi(\boldsymbol{y})\,.$$

Hence, the main terms on the right-hand side in Eq. (3.2) and (3.10) for $\kappa \to 0$ are given by the renormalized stochastic integrals

$$\hat{\lambda}(t) = \frac{1}{\nu t}\int_0^t L(n_r)\,\mathrm{d}n_r = \frac{1}{\nu t}\,\hat{n}_t[L] \tag{5.3}$$

of the operator-valued stochastic functions $L(t,\tau) = L(n_t(\tau))$ with respect to the numerical process $n_t(\tau)$ that has the Poisson probability distribution (3.5) on Γ_∞.

To pass to the large number limit $\nu \to \infty$ in (5.4) for an arbitrary operator L in $L^2(\Lambda)$, we need to use the quantum stochastic representation [22] of the integral (5.4) in the Fock space $\mathcal{F}$ over $L^2(\mathbb{R}_+ \times \Lambda)$. The space $\mathcal{F}$ is defined as the $L^2(\Upsilon)$-space of all square integrable functions $\phi : \Upsilon \to \mathbb{C}$, $\|\phi\|^2 = \int_\Upsilon |\phi(\upsilon)|^2 \lambda(\mathrm{d}\upsilon) < \infty$ of time ordered finite sequences $\upsilon = (y_1, \ldots, y_n)$, $y = (t, \mathbf{y})$ identified with subsets $\upsilon \subset \mathbb{R}_+ \times \Lambda$ of finite cardinality $|\upsilon| = 0, 1, 2, \ldots$. The measure $\lambda(\mathrm{d}\upsilon)$ on the union

$$\Upsilon = \sum_{n=0}^{\infty} \Upsilon(n)$$

of the disjoint subsets $\Upsilon(n) = \{\upsilon \in \Upsilon : |\upsilon| = n\}$ is given as the sum

$$\lambda(A) = \sum_{n=0}^{\infty} \lambda(\Upsilon(n) \cap A)$$

of the product $\lambda(\mathrm{d}\upsilon) = \prod_{y \in \upsilon} \mathrm{d}y$ of measures $\mathrm{d}y = \mathrm{d}t\, \mathrm{d}\lambda$ on $\mathbb{R}_+ \times \Lambda$ such that

$$\|f\|^2 = \sum_{n=0}^{\infty} \iint_{0 \le t_1 < \ldots \le t_n < \infty} |f(y_1, \ldots, y_n)|^2 \prod_{i=1}^{n} \mathrm{d}y_i \,.$$

Let $N_t[L]$ denote the numerical integral as an operator

$$(N_t[L]\phi)(\tau, \boldsymbol{y}) = \sum_{n=1}^{n_t(\tau)} L(n)\varphi(\tau, \boldsymbol{y}) = \hat{n}_t[L](\tau)\phi(\tau, \boldsymbol{y}) \tag{5.4}$$

in the Fock space $\phi \in L^2(\Upsilon)$. This integral represents the operator-valued stochastic integral (5.2) by pointwise multiplication of the functions $\phi(\upsilon) = \phi(t, \boldsymbol{y})$ of $\upsilon \in \Upsilon$, by $\hat{n}_t[L](\tau)$, which is considered as the function of a finite sequence $\tau \in \Gamma$ because of its independence of $\tau_{[t} = \{t_n \ge t\}$. In order to obtain the initial probability measure $\mathrm{P}_0(\mathrm{d}\upsilon) = \pi(\mathrm{d}\tau)\mu_0^{\infty}(\mathrm{d}\boldsymbol{y})$ on Υ_∞ induced by an initial Fock vector $\phi_0 \in L^2(\Upsilon)$, we need an isomorphic transformation of (5.5)

$$\hat{N}_t[L] = N_t[L] + \sqrt{\nu}(A_t[f_0^\dagger L] + A_t^\dagger[L f_0]) + \nu t f_0^\dagger L f_0 \tag{5.5}$$

which can be locally performed by a unitary transformation

$$\hat{N}_t[L] = U_s^\dagger N_t[L] U_s \,, \quad U_s = \exp\{\sqrt{\nu}(A_s^\dagger[f_0] - (A_s[f_0^\dagger])\}$$

for every $t < s$. Here $A_t^\dagger[f]$ and $A_t[f^\dagger]$ are the creation and annihilation integrals of $f \in L^2(\Lambda)$, $f^\dagger \in L^2(\Lambda)^*$, given by the operators

$$\Big(A_t^\dagger[f]\phi\Big)(\upsilon) = \sum_{y \in \upsilon_t} f(\mathbf{y})\phi(\upsilon \backslash y),$$
$$\Big(A_t[f^\dagger]\phi\Big)(\upsilon) = \int_{[0,t)\times\Lambda} f(\mathbf{y})^* \phi(\upsilon \sqcup y)\mathrm{d}y$$

in the Fock space $L^2(\Upsilon)$, where $\upsilon \backslash y$ means the sequence $\upsilon \in \Upsilon$ with deleted $y = (r, \mathbf{y})$, $r < t$, and $\upsilon \cup y$ means the sequence $\upsilon \in \Upsilon$ with an additional element $y \notin \upsilon$. The characteristic functional of the stochastic operators $\hat{n}_t[L]$ with respect to the initial state-vector $\varphi_0 \simeq f_0^\infty$ in $\mathcal{E}_\infty$ and the Poisson probability measure (3.5) is now given simply by the vacuum expectation

$$\int_{\Gamma^\infty} (\varphi_0, e^{\mathrm{i}\hat{n}_t[L]}\varphi_0)\pi_0(\mathrm{d}\tau) = \langle \delta_\phi, e^{\mathrm{i}\hat{N}_t[L]}\delta_\phi \rangle \,,$$

where $\delta_\phi(\upsilon) = 1$ if $\upsilon = \emptyset$; otherwise, $\delta_\phi(\upsilon) = 0$.

The corresponding representation $\hat{l}(t) = \frac{1}{\nu t}\,\hat{N}_t[L]$ for (5.4) helps us immediately obtain the quantum large number limit

$$\lim_{\nu\to\infty} \frac{1}{\nu t}\,\hat{N}_t[L] = f_0^\dagger L f_0 \hat{1}$$

as the mean value $l_0 = (f_0, Lf_0) \equiv f_0^\dagger L f_0$ of a single-bubble operator with respect to an initial wave packet $f_0 \in L^2(\Lambda)$. This gives the following:

Proposition 4 *The macroscopic limit*

$$\mathrm{d}\psi(t) + \frac{\mathrm{i}}{\hbar}\,H_0\psi(t)\,\mathrm{d}t = \frac{\mathrm{i}}{\hbar}\,\gamma(\mathbf{R}\otimes\mathbf{p}_0\hat{1})\psi(t)\,\mathrm{d}t$$

of the generalized Schrödinger equation (3.2) turns out to be nonsingular with an additional potential $-\gamma\mathbf{p}_0\mathbf{R}$ corresponding to the mean momentum $\mathbf{p}_0 = (f_0, \mathbf{P}f_0)$ of a bubble in the initial state f_0. It coincides with the large number limit $\nu\to\infty$ of the reduction equation (3.10) under $\kappa = -\gamma/\nu$.

Proof. Indeed, the mean field dynamics preserves the product structure $\psi(t,\upsilon) = \eta(t)\phi_0(\upsilon)$ of the initial product-vector $\psi_0 = \eta\otimes\phi_0$ on the Fock component $\phi_0 \in \mathcal{F}$ because it is identical by virtue of $H_0 = H\otimes\hat{1}$. But unexpectedly (compare with [19]) the large number limit

$$\mathrm{d}\chi(t) + \frac{\mathrm{i}}{\hbar}\,H\chi(t)\,\mathrm{d}t = \frac{\mathrm{i}}{\hbar}\,\gamma\mathbf{p}_0\mathbf{R}\chi(t)\,\mathrm{d}t \qquad (5.6)$$

of the stochastic nonunitary equation (3.10) corresponds to the same unitary dynamics $\eta(t) = U(t)\eta = \chi(t)$ of the particle state vector if $\chi(0) = \eta$, since

$$\frac{1}{\nu t}\,n_t\!\left(\frac{f_0'}{f_0}\right) \to (f_0, L'f_0) = \int f_0(\mathbf{y})^* f_0'(\mathbf{y})\,\mathrm{d}\mathbf{y} = \frac{\mathrm{i}}{\hbar}\,\mathbf{p}_0\,.$$

■

Note that the macroscopic limits for the M-particle system also give essentially the same type of unitary evolutions in the large space $\mathcal{H}^M\otimes\mathcal{F}$ and in the reduced space $\mathcal{H}^M$. To get this correspondence, one has only to replace the measure (3.5) on Γ_∞ for Eq. (4.2) by the product measure $\pi_0^{\otimes M}$ on Γ_∞^M and for Eq. (4.10) by the induced measure (4.7) on Γ_∞, taking $M\nu$ instead of ν in (5.5). This means that the mixing property of the reduced equation (4.10) vanishes in the mean field approximation for the bubble system.

Let us now pay attention to the fluctuations with respect to the obtained large number limits. Such fluctuations might appear for $-\kappa = \gamma/\nu \to 0$ in the large time scale $t \sim 1/\kappa$. We can get these fluctuations without rescaling the time t if we assume that $p_0 = 0$ and $-\kappa = \gamma/\sqrt{\nu}$, so that we have to take into account also the κ^2-terms in (5.1).

It follows from the Fock space representation (5.6) that the quantum central limit

$$\lim_{\nu\to\infty} \frac{l}{\sqrt{\nu}}\,\hat{N}_t[L] = A_t[f_0^\dagger L] + A_t^\dagger[Lf_0]$$

exists for any single-bubble operator L with zero mean value $(f_0, Lf_0) = 0$. We first apply this central limit theorem to the right-hand side in (3.2) represented in $\mathcal{H} \otimes \mathcal{F}$ as

$$\hat{n}_t[S - I] = \mathrm{i}\frac{\gamma}{\hbar}\frac{1}{\sqrt{\nu}}\hat{N}_t[\mathbf{R} \otimes \mathbf{P}] - \frac{1}{2}\left(\frac{\gamma}{\hbar}\right)\frac{1}{\nu}N_t[(\mathbf{R} \otimes \mathbf{P})^2] + \dots .$$

This yields $\lim \frac{1}{\sqrt{\nu}}\hat{N}_t[\mathbf{P}] = \hat{\mathbf{u}}_t$, $\lim \frac{1}{\nu}\hat{N}_t[P_\alpha P_\beta] = (P_\alpha f_0, P_\beta f_0)t$, and

$$\mathrm{d}\psi(t) + K_0\psi(t)\,\mathrm{d}t = \frac{\mathrm{i}}{\hbar}\,\gamma(\mathbf{R} \otimes\, \mathrm{d}\hat{\mathbf{u}}_t)\psi(t)\,. \tag{5.7}$$

Here $K_0 = K \otimes \hat{1}$, $K = \frac{\mathrm{i}}{\hbar}\,H + \frac{1}{2}\left(\frac{\gamma}{\hbar}\right)^2 \mathbf{R}\boldsymbol{\sigma}^2\mathbf{R}$, $\boldsymbol{\sigma}^2 = [\sigma^2_{\alpha\beta}]$,

$$\hat{\mathbf{u}}_t = A_t[f_0^\dagger \mathbf{P}] + A_t^\dagger[\mathbf{P}f_0] = 2\Re A_t^\dagger[\mathbf{P}f_0]$$

is a Fock space representation of the Wiener vector process $\boldsymbol{u} : t \mapsto \mathbf{u}_t$ with the correlations $\sigma^2_{\alpha\beta} = \hbar^2(\partial_\alpha f_0, \partial_\beta f_0)$, $\alpha, \beta = 1, \dots, d$, defined by the momentum operators $\mathbf{P}$ due to the quantum stochastic multiplication formula [21, 22]

$$\mathrm{d}\hat{u}_\alpha\,\mathrm{d}\hat{u}_\beta = \mathrm{d}A_t[f_0^\dagger P_\alpha]\,\mathrm{d}A_t^\dagger[P_\beta f_0] = f_0^\dagger P_\alpha P_\beta f_0\,\mathrm{d}t\,.$$

The central limit equation (5.8) for the unitary evolution of the coupled system turns out to be a stochastic Schrödinger-Itô equation of diffusive type driven by the Wiener process $\boldsymbol{u} = \{\mathbf{u}_t : t \in \mathbb{R}_+\}$. The same conclusion obviously holds for the M-particle system driven by M independent Wiener processes $\hat{\mathbf{u}}_t(k)$ identical to $\hat{\mathbf{u}}_t$, $k = 1, \dots, M$:

$$\mathrm{d}\psi^M(t) + K_0^M\psi(t)\,\mathrm{d}t = \frac{\mathrm{i}}{\hbar}\,\gamma\left(\sum_{k=1}^{M}\mathbf{R}(k) \otimes\, \mathrm{d}\hat{\mathbf{u}}_t(k)\right)\psi^M(t)\,,$$

where $K^M = \frac{\mathrm{i}}{\hbar}\,H^M + \frac{1}{2}\left(\frac{\gamma}{\hbar}\right)^2 \sum_{k=1}^M \mathbf{R}(k)\boldsymbol{\sigma}^2\mathbf{R}(k)$, $K_0^M = K^M \otimes \hat{1}$.

We now prove that the application of the central limit theorem to the reduction equation (3.10) yields an essentially different type of the stochastic evolution, originally derived in [21] by quantum calculus method.

Theorem 4 *The central limit $\nu = (\gamma/\kappa)^2 \to \infty$ of the stochastic wave equation (3.10) has the following nonunitary diffusive type*

$$\mathrm{d}\chi(t) + K\chi(t)\,\mathrm{d}t = \gamma\mathbf{R}\chi(t)\,\mathrm{d}\hat{\mathbf{v}}_t\,. \tag{5.8}$$

Here $\mathbf{v}_t$ is a complex Wiener vector-process with the Fock-space representation

$$\hat{\mathbf{v}}_t = A_t[f_0^\dagger L'] + A_t^\dagger[L'f_0] = \Re A_t^\dagger[(\mathbf{w}_0 + \bar{\mathbf{w}}_0)f_0] + \mathrm{i}\Im A_t^\dagger[(\mathbf{w}_0 - \bar{\mathbf{w}}_0)f_0] \tag{5.9}$$

given by the complex osmotic velocity $\mathbf{w}_0(\mathbf{y}) = \partial \ln f_0(\mathbf{y})$, $\partial = (\partial_1, \dots, \partial_d)$ of a single bubble, and the operator K is the same as in (5.8).

Proof. Indeed, the central limit of the right-hand side in (3.10) with $\kappa = -\gamma/\sqrt{\nu}$ yields

$$\begin{aligned}\lim \mathrm{d}n_t[G-I] &= \gamma \mathbf{R} \lim \frac{1}{\sqrt{\nu}}\, \mathrm{d}n_t\left(\frac{f_0'}{f_0}\right) + \frac{1}{2}\,\gamma^2 \mathbf{R} \lim \frac{1}{\nu}\, \mathrm{d}n_t\left(\frac{f_0''}{f_0}\right)\mathbf{R} \\ &= \gamma \mathbf{R} \lim \frac{1}{\sqrt{\nu}}\, \mathrm{d}\hat{N}_t[L'] + \frac{1}{2}\,\mathbf{R} \lim \frac{1}{\nu}\, \mathrm{d}\hat{N}_t[L'']\mathbf{R} \\ &= \gamma \mathbf{R}\, \mathrm{d}\hat{\mathbf{v}}_t - \frac{1}{2}\left(\frac{\gamma}{\hbar}\right)^2 \mathbf{R}\boldsymbol{\sigma}^2\mathbf{R}\, \mathrm{d}t\,,\end{aligned}$$

where the representation (5.6) is used for

$$\hat{n}_t[L'] = n_t\left(\frac{f_0'}{f_0}\right) \quad \text{and} \quad \hat{n}_t[L''] = n_t\left(\frac{f_0''}{f_0}\right)$$

so that $f_0^\dagger \partial_\alpha \partial_\beta f_0 = -(\partial_\alpha f_0, \partial_\beta f_0)$ are the matrix elements of $\lim \frac{1}{\nu t}\, \hat{N}_t[L''] = f_0^\dagger L'' f_0$. The linear stochastic equation (5.9) obtained has a unique solution $\chi(t,\boldsymbol{v}) = T(t,\boldsymbol{v})\eta$ for a given $\chi(0,\boldsymbol{v}) = \eta \in \mathcal{H}$ which is not normalized $\|\chi(t,\boldsymbol{v})\| \neq 0$ for every Wiener trajectory $\boldsymbol{v} : t \mapsto \mathbf{v}_t$ but is normalized in the mean square sense $\int \|\chi(t,\boldsymbol{v})\|^2 \mathrm{P}_0(\mathrm{d}\boldsymbol{v}) = 1$ with respect to the Gaussian probability measure P_0 of $\boldsymbol{v} = \{\mathbf{v}_t : t \in \mathbb{R}_+\}$. The measure P_0 is defined by the zero mean values of $\mathbf{v}_t$ and by the table

$$\begin{aligned}\mathrm{d}\hat{v}_\alpha\, \mathrm{d}\hat{v}_\beta &= \mathrm{d}A_t[f_0^\dagger L_\alpha']\mathrm{d}A_t^\dagger[L_\beta' f_0] = f_0^\dagger L_\alpha' L_\beta' f_0\, \mathrm{d}t \\ \mathrm{d}\hat{v}_\alpha^*\, \mathrm{d}\hat{v}_\beta &= \mathrm{d}A_t[f_0^\dagger L_\alpha'^\dagger]\mathrm{d}A_t^\dagger[L_\beta' f_0] = f_0^\dagger \hat{L}_\alpha'^\dagger L_\beta' f_0\, \mathrm{d}t\end{aligned}$$

of commuting multiplications

$$\mathrm{d}\hat{v}_\alpha\, \mathrm{d}\hat{v}_\beta = \mathrm{d}\hat{v}_\beta\, \mathrm{d}\hat{v}_\alpha\,, \quad \mathrm{d}\hat{v}_\alpha\, \mathrm{d}\hat{v}_\beta^* = \mathrm{d}\hat{v}_\beta^*\, \mathrm{d}\hat{v}_\alpha\,.$$

But the reduction noise $\hat{\mathbf{v}}_t$ obtained in the Fock space representation does not commute with the real Wiener process $\hat{\mathbf{u}}_t = \hat{\mathbf{u}}_t^*$ in (5.8),

$$\begin{aligned}\mathrm{d}\hat{v}_\alpha\, \mathrm{d}\hat{u}_\beta &= \mathrm{d}A_t[f_0^\dagger]\, \mathrm{d}A_t^\dagger[P_\beta f_0] = f_0^\dagger L_\alpha' P_\beta f_0\, \mathrm{d}t \\ \mathrm{d}\hat{u}_\alpha\, \mathrm{d}\hat{v}_\beta &= \mathrm{d}A_t[f_0^\dagger]\, \mathrm{d}A_t^\dagger[L_\beta' f_0] = f_0^\dagger P_\alpha L_\beta' f_0\, \mathrm{d}t\end{aligned}$$

if $g_{\alpha\beta}(\mathbf{y}) \equiv \partial_\alpha \partial_\beta \ln f_0(\mathbf{y}) \neq 0$, since

$$[\mathrm{d}\hat{v}_\alpha, \mathrm{d}\hat{u}_\beta] = f_0^\dagger [L_\alpha', P_\beta] f_0\, \mathrm{d}t = \frac{\hbar}{\mathrm{i}}\,(f_0, g_{\alpha\beta} f_0)\, \mathrm{d}t\,.$$

■

In the same way one can obtain the continuous reduction equation

$$\mathrm{d}\rho^M(t) + (K\rho^M(t) + \rho^M(t)K^\dagger)\mathrm{d}t = \left(\frac{\gamma}{\hbar}\right)^2 \sum_{k=1}^{M} \mathbf{R}(k)\boldsymbol{\sigma}^2 \rho^M(t)\mathbf{R}(k)\, \mathrm{d}t + \gamma(\mathrm{d}\mathbf{w}_t \mathbf{R}\rho^M(t) + \rho^M(t)\mathbf{R}\, \mathrm{d}\mathbf{w}_t^*), \quad (5.10)$$

$\mathbf{R} = \frac{1}{M}\sum_{k=1}^{M}\mathbf{R}(k)$, where the operator $K = K^M$ is the same as in the equation for the unitary evolution of the M-particle system coupled with the bubbles. The derived stochastic equation for the M-particle density operator $\rho^M(t,\boldsymbol{w})$ normalized in the mean is driven by the complex Wiener process $\mathbf{w}_t = \mathbf{v}_t^M$ having the same multiplication table as the process $\sqrt{M}\mathbf{v}_t$ in (5.9). The diffusive type equation (5.10) as (4.10) also has the mixing property. This was also derived in [5] by operational method.

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